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A model for ripple instabilities in granular media

O. Terzidis[a], P. Claudin, and J.-P. Bouchaud

Service de Physique de l'État Condensé, C.E. Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France

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Abstract. We extend the model of surface granular flow proposed in [1] to account for the effect of an external 'wind', which acts as to dislodge particles from the static bed, such that a stationary state of flowing grains is reached. We discuss in detail how this mechanism can be described in a phenomenological way, and show that a flat bed is linearly unstable against ripple formation in a certain region of parameter space. We focus in particular on the (realistic) case where the migration velocity of the instability is much smaller than the grains' velocity. In this limit, the full dispersion relation can be established. We relate the critical wave vector to the mean hopping length and to the ratio of the flight time to the 'stick' time. We provide an intuitive interpretation of the instability.

PACS. 83.70.Fn Granular solids – 81.05.Rm Porous materials; granular materials – 47.20.-k Hydrodynamic stability

1 Motivation

Common observations suggest that flat sand surfaces can become unstable when subjected to moving air or water. After some time regular patterns appear, as can be observed on desert dunes, underwater sand, 'dry' snow, *etc.* These patterns resemble surface waves; however their physics is completely different since in the case of sand there is no surface tension. Following Bagnold ([2], chap. 11) these patterns can be classified into ripples, ridges and dunes. The repetition distance of ridges varies with time, whereas ripples exhibit a stationary wavelength after some transient. Early qualitative arguments by Bagnold [2] suggested that the ripple wavelength λ is related to the typical path length of the blown grains, called the 'saltation length' ξ. A more quantitative 'two-species' model was proposed by Anderson [3], which describes the coupling between the moving grains and the static bed. Such a model predicts that a flat surface is unstable for all wavelengths, with a faster growing mode indeed comparable to the typical jump length of the grains. However, this model is incomplete: while the dynamics of the static bed is treated exactly, the description of the moving phase is highly simplified. Alternatively, there are also several numerical models for ripple formation [4]. In this paper, we extend Anderson's theoretical model of ripple formation, by adapting the phenomenological equations for surface flow introduced in [1] in the context of avalanches, and further discussed in [5–9].

[a] e-mail: terzidis@spec.saclay.cea.fr

It is worth recalling, after Bagnold [2], some basic facts about the motion of the grains and the formation of these patterns: (i) There are two qualitatively distinct transport mechanisms for the grains, saltation and surface creep[1]. The trajectories of grains in saltation is determined by the velocity profile of the wind, the air friction limiting the grain velocity and by the initial energy of the grain when first expelled from the sand bed. One of the characteristic features of the path is the flat angle of incidence which varies between 10° and 15°. (ii) The saltation has two effects on the surface: it either rebounds and/or ejects grains leading to a new saltation or it produces surface creep. There is however no sharp boundary between these two processes, since the energy of the ejected grains varies continuously. Both saltation and creep lead to a net flow of grains in the direction of the wind. (iii) The time scale of ripple formation is much larger than that of saltation. (iv) The migration velocity of the ripples is much smaller than a mean transport velocity (averaged over saltation and creep).

2 A 'two-species' model with wind

The phenomenological approach we consider in the following is based on the observation that two different species of grains enter the problem: moving grains and grains at rest. We will not distinguish between grains in saltation

[1] To which one should also add 'suspension', corresponding to very small grains flying high in the air.

and creep, but introduce an appropriately averaged quantity describing grains that are convected by either of the two mechanisms, which we call the moving grain density $R(x,t)$ [2], where x is the coordinate in the direction of the wind and t the time. (We will assume that the problem is translationally invariant in the direction transverse to the wind; see [9] for an extension to two dimensions). The grains at rest contribute to the local height $h(x,t)$ of the static bed. The dynamical equations for R and h read, in the hydrodynamical (long wavelength) limit:

$$\begin{aligned}\partial_t R &= -V\partial_x R + D_1\partial_x^2 R + \Gamma[R,h]\\ \partial_t h &= -\Gamma[R,h]\end{aligned} \tag{1}$$

where V and D_1 are the average velocity of the grains and the dispersion constant, related to the fact that grains do not all move with the same velocity[3]. Note that V can be much smaller than the saltation velocity if the predominant mechanism is reptation. Γ describes the rate with which a grain at rest is converted into a moving grain (or vice versa) and depends both on R and on the local surface profile h. For simplicity we have defined R to have the same dimension as h, and it can be thought of as the width of grains which has been removed from the static bed. Correspondingly, Γ has the dimension of a velocity. The construction of Γ is based on phenomenological arguments [1], and encodes different physical processes:

- Due to the presence of wind, grains can be 'spontaneously' ejected from the surface, even in the absence of already moving grains. The rate at which this occurs depends on the local wind velocity (or rather velocity gradient at the surface). Since the wind velocity tends to be larger when the local slope is facing the wind, we write:

$$\Gamma_{sp} = \alpha_0 + \alpha_1\partial_x h - \alpha_2\partial_x^2 h \tag{2}$$

where the coefficients α_i are positive or zero. We have also included the second derivative contribution with a minus sign, since grains are harder to dislodge in troughs than at the top of a crest. Note that all these coefficients are expected to depend on the external wind velocity. In particular, as shown by Bagnold himself, the coefficient α_0 is only non-zero above a certain critical wind velocity, noted V^*_{fluid}.

- When hitting the ground, a moving grain can either be captured or transfer a part of its kinetic energy to other static grains and provide new moving particles. The rate at which both these process occur is proportional to R (at least for small enough R – see below), and also depends on the wind velocity and on the local slope.

[2] In principle, one should consider a density $R(x,v,t)$ which depends on the instantaneous velocity of the grains. $R(x,t)$ is the average of $R(x,v,t)$ over all velocities.

[3] These terms can be understood, more generally, as the long-wavelength limit of a more general non-local convection term of the kind $\int K(x-x')R(x',t)\,dx'$.

Table 1. Summary of the different coefficients and their signs. Only γ_0 and γ_1 can be negative, which corresponds to the situations where capture dominates (low wind, or under water).

transport coefficients	spontaneous processes	stimulated processes	saturation coefficient
$V>0, D_1>0$	$\alpha_i>0$	$\gamma_i \quad (\gamma_2>0)$	$\beta>0$

This suggests to write the *stimulated* conversion rate as:

$$\Gamma_{st} = R[\gamma_0 + \gamma_1\partial_x h - \gamma_2\partial_x^2 h]. \tag{3}$$

The sign of γ_0 depends on the strength of the wind; for small wind velocity, one expects capture to be more important than emission, and thus that $\gamma_0 < 0$. As again shown by Bagnold, a localized source of moving grains tends to die away when the wind velocity is less than a certain $V^*_{impact} < V^*_{fluid}$, whereas a steady saltation is found for larger velocities, suggesting that $\gamma_0 > 0$ for $V > V^*_{impact}$. In this case, however, it is easy to see that R increases exponentially, and that higher order terms are needed to describe the stationary situation. One can think of several non-linear effects: for example, collision between flying grains leads to dissipation and hence to a poorer efficiency of the impacts on the static bed. Also, the presence of a layer of moving grains screens the hydrodynamical flow, which in turn reduces the energy transfer between the wind and the saltating grains. To leading order, it is reasonnable to describe these effects by adding a term[4] $-\beta R^n$ with $n=2$ in Γ_{st}. The following linear stability analysis is however independent of n. If trapping dominates (as is the case for under water ripples) one expects $\gamma_1 < 0$ because more grains are captured on the slope facing the convective flow. For the same reason, if stimulated emission dominates, as is the case for wind blown sand, one expects that $\gamma_1 > 0$, since impacts induce more flying grains. Finally, γ_2 is positive since, again, grains are easier to dislodge at the top of a bump.

The total conversion rate Γ is obtained as the sum of Γ_{sp} and Γ_{st}, while the model proposed in [1] did not contain the wind induced contribution proportionnal to α, nor the non-linear term. The equation for h thus reads:

$$\begin{aligned}\partial_t h = &-(R\gamma_0+\alpha_0) + \beta R^n - (R\gamma_1+\alpha_1)\partial_x h\\ &+(R\gamma_2+\alpha_2)\partial_x^2 h.\end{aligned} \tag{4}$$

Note that the coefficients α_i, γ_i (summarized in Tab. 1) actually only enter through the combination $\alpha_i + R\gamma_i$. The gradient term can be interpreted as a translation of the surface profile with time, at velocity $W = \alpha_1 + R\gamma_1$. The sign of W is not fixed *a priori*: the direct action of the wind (α_1) is indeed to erode grains from the windward slope of a bump and transport them in the direction of the wind. The other contribution ($R\gamma_1$), however, moves the bumps 'backwards' whenever the $\gamma_1 < 0$: grains are

[4] In principle, the dependence of V on R should also be taken into account. We do not consider this here, since this does not affect the linear instability analysis.

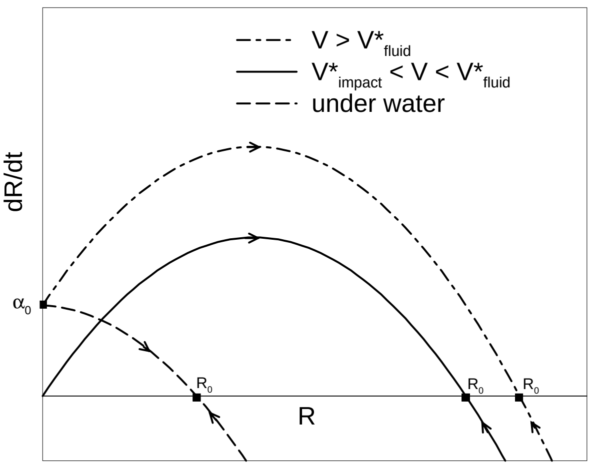

Fig. 1. Stability diagram, showing dR/dt as a function of R in an homogeneous situation for $n = 2$. The case $\gamma_0 > 0$ corresponds to blown sand with $V > V^*_{impact}$, where stimulated emission is very efficient, and where R_0 can be non-zero even if $\alpha_0 = 0$ (*i.e.* when $V < V^*_{fluid}$). The situation where capture dominates ($\gamma_0 < 0$) is probably relevant for sand under water.

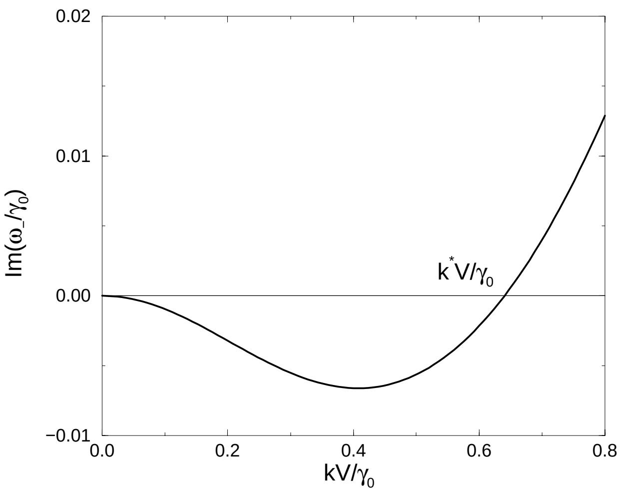

Fig. 2. Rescaled damping rate as a function of the rescaled wave vector. The plot shows data for $\eta = 0.1$, $\omega_0 D_1 = V^2$ and $D_2/D_1 = 0.1$.

effectively deposited on the windward slope, contributing to a translation of the bump against the wind. (A similar discussion can be found in [1,8,9].)

Note that the above set of equations is non-linear, so that non-trivial dynamics is expected. Some essential features of the model can be investigated by linearizing the system in the vicinity of the situation where the surface is flat ($h_0 = 0$). The moving grain density is given by the solution of the equation $\alpha_0 + \gamma_0 R_0 - R_0^n = 0$. For the case $n = 2$ (see Fig. 1) it reads:

$$R_0 = \frac{1}{2\beta}\left[\gamma_0 + \sqrt{\gamma_0^2 + 4\alpha_0\beta}\right]. \quad (5)$$

3 Stability analysis

We will perform a stability analysis, *i.e.* investigate whether a small perturbation is amplified or dies out with time. Therefore we consider $R = R_0 + \bar{R}$, $h = h_0 + \bar{h}$ and neglect second order terms of the kind $\bar{R}\bar{h}$, $\bar{R}^2$ and $\bar{h}^2$. For simplicity of notation we drop the bars; the linearized equations then read

$$\begin{aligned} \partial_t R &= -\omega_0 R - V\partial_x R + D_1\partial_x^2 R + W\partial_x h - D_2\partial_x^2 h + ... \\ \partial_t h &= \omega_0 R - W\partial_x h + D_2\partial_x^2 h + ... \end{aligned} \quad (6)$$

with an effective velocity $W = \alpha_1 + R_0\gamma_1$ and an effective diffusion constant $D_2 = \alpha_2 + R_0\gamma_2 > 0$. ω_0 is equal to $n\beta R_0 - \gamma_0$; we will assume that it is *positive* (which is always true for $n = 2$ where $\omega_0 = \sqrt{\gamma_0^2 + 4\alpha_0\beta}$). A Fourier analysis of the linearized equations leads to

$$\begin{pmatrix} -i\omega - \omega_0 - ikV - k^2 D_1 & ikW + k^2 D_2 \\ \omega_0 & -i\omega - ikW - k^2 D_2 \end{pmatrix} \begin{pmatrix} \widetilde{R} \\ \widetilde{h} \end{pmatrix} = 0 \quad (7)$$

where the tilde denotes the Fourier transforms. This system has a non-trivial solution if the determinant of the above matrix is zero, leading to the relation

$$\omega^2 + \omega(a + ib) + (c + id) = 0. \quad (8)$$

The coefficients read

$$\begin{aligned} a &= (V + W)k \\ b &= -\left[\omega_0 + (D_1 + D_2)k^2\right] \\ c &= VWk^2 - D_1 D_2 k^4 \\ d &= -(D_1 W + D_2 V)k^3; \end{aligned} \quad (9)$$

they are functions of the wave vector k and of the system's parameters (V, W, D_1, D_2, ω_0).

Equation (8) establishes a dispersion relation $\omega(k)$ with two branches corresponding to the two solutions of the quadratic equation, where ω has to be considered as a complex variable. (Writing down the corresponding equations for the real and the imaginary part of ω leads to quartic equations.) In the context of a stability analysis we are interested in the imaginary part of $\omega(k)$: as long as it is positive $e^{i\omega t}$ will decay exponentially, while a negative imaginary part does lead to an instability. This imaginary part is given by:

$$\begin{aligned} 2\mathrm{Im}(\omega_\pm) = -b \pm \frac{1}{\sqrt{2}}\Big[&-(a^2 - b^2 - 4c) \\ &+ \left[(a^2 - b^2 - 4c)^2 + (2ab - 4d)^2\right]^{1/2}\Big]^{1/2} \end{aligned} \quad (10)$$

which is a function of k. A critical wave vector k^* can be defined such that $\mathrm{Im}(\omega)$ is exactly zero, which leads to $d^2 - abd + b^2 c = 0$. Inserting the explicit expressions (9), one finds a cubic equation for k^{*2}. Whenever this equation admits a positive solution, there will be a finite band of wave vectors $[0, k^*]$ which are unstable (see Fig. 2).

Note that for vanishing diffusion constants all wavelengths are actually unstable. However, the curve $\mathrm{Im}(\omega(k))$ still has a minimum, corresponding to the fastest growing mode.

It is instructive to study the asymptotic behaviour of the functions $\mathrm{Im}(\omega_-)$. One finds:

$$\mathrm{Im}(\omega_-) = \begin{cases} -\frac{\eta V^2}{\omega_0}\, k^2 + ... & \text{for } k \ll \omega_0/V \\ D_2 k^2 + ... & \text{for } k \gg \omega_0/V \end{cases}. \quad (11)$$

with $\eta = W/V$. The transport velocity V (by convention) and the diffusion constants are positive; the main control parameter remaining is the relative migration velocity η. One can show that for $0 < \eta < 1$ there is indeed a band of instable wave vectors (one can see from the asymptotic solutions that the sign changes for large k's). The situation where $\eta < 0$ is stable and $\eta > 1$ (*i.e.* a bump moving faster than the transport velocity) does not seem physical. The second branch $\mathrm{Im}(\omega_+)$ is always positive and is thus of no importance for our stability considerations.

Following the intuition that the ripples move much more slowly than the grains are transported, we will assume in the sequel that $0 < \eta \ll 1$, which we attribute to the fact that the α coefficients are small compared to V. Since $D_2 \propto \alpha$, this suggests that the diffusion constants D_1 and D_2 are in the same ratio, so we write: $D_2 = \delta\eta D_1$, where δ is of the order of one. These assumptions make it possible to simplify the algebra and to find the solution:

$$\mathrm{Im}(\omega_-) = \eta\, \frac{k^2\left[-\omega_0 V^2 + D_1\delta(\omega_0 D_1 + V^2)k^2 + \delta D_1^3 k^4\right]}{\omega_0^2 + k^2(2\omega_0 D_1 + V^2) + k^4 D_1^2} + o(\eta^2). \quad (12)$$

which is plotted in Figure 2. As we discuss now, three relevant facts can be verified with this formula: (i) the critical wave vector is of the order of the inverse mean hopping length, (ii) the ripple velocity is of the order of ηV and grows with the wavevector (*i.e.* small ripples are faster than large ones) and (iii) the time scale of ripple formation is much larger than the saltation time scale.

Let us first give some arguments for (i). Since the saltation trajectories result from some random initial vertical velocity of the grains, the hopping lengths will also be random, with both short jumps (actually corresponding to creep) and long jumps (corresponding to saltation). It is reasonable to assume that the width of the hopping length distribution is of the same order as its mean ξ (a similar assumption is discussed in [3]). In this situation, the 'Péclet' number defined as $Pe = V\xi/D_1$ is of order one: convective and diffusive effects are of the same order of magnitude. In the case where the jump length distribution is sharply peaked around ξ, one would rather have $Pe \gg 1$.

Defining $\xi = V\tau$, where τ is the typical flying time between two collisions with the static bed, one finds that the zero of (12) is located at:

$$k_*^2 = \frac{Pe}{2\xi^2}\left\{\sqrt{(\omega_0\tau + Pe)^2 + 4\omega_0\tau/\delta} - (Pe + \omega_0\tau)\right\}. \quad (13)$$

In the case where the sticking probability is large, it is reasonnable to assume that $\omega_0\tau \sim 1$, thereby leading to $k^* \sim \xi^{-1}$ for Pe ~ 1. On the other hand, for weakly dissipative collisions (hard grains) or strong wind one expects that $\omega_0\tau \ll 1$, leading to unstable wavelengths $\sim \xi\sqrt{\delta/\omega_0\tau}$ much larger than the mean hopping length.

The ripple velocity is given by the corresponding dispersion relation, *i.e.* the real part of $\omega(k)$. One finds:

$$2\mathrm{Re}(\omega_-) = \eta\, \frac{k^3 V\left[V^2 + \omega_0 D_1(1+\delta) + D_1^2 k^2\right]}{\omega_0^2 + k^2(2\omega_0 D_1 + V^2) + k^4 D_1^2} + o(\eta^2). \quad (14)$$

The formula shows that for $k \sim k^*$, both phase and group velocities are, as expected, of the order of ηV; furthermore, it can be seen that the group velocity increases with k, thus establishing (ii).

Finally knowing the fastest growing wave vector, one finds that the ripple formation time t_{ripple} (determined by the depth of the minimum in Fig. 2) is a factor $1/\eta$ larger than say $1/\omega_0$ or τ, *i.e.* that ripple formation occurs on much slower time scales than any microscopic process. The ratio of formation time and microscopic time scales should indeed be roughly the same as that between migration and convection velocity (iii).

4 Physical discussion and open questions

Let us finally give an intuitive interpretation of the instability. Imagine a flat surface with a finite number of moving grains above it (*i.e.* the stationary solution). Now imagine a small perturbation of this situation, say a small hump. The term $\partial_t R \sim \partial_x h$ in the linearized equations (6) increases locally the concentration of the moving grains thus producing a 'cloud' at the windward side of the hump. This cloud is convected with the velocity V and after a time unit of $1/\omega_0$ the cloud has moved a distance ξ where the cloud starts to 'rain' (*i.e.* moving grains are converted into grains at rest). If the position of the hump has in the same time moved (albeit at a much smaller velocity) in the same direction, its height will increase, leading to an instability. (Conversely, if the bump moves backward – *i.e.* if $W < 0$ – the 'rain' will rather fill the hole and smear out the bump.) The presence of the diffusive processes counterbalances the amplification for small distances and some optimum wavelength of the order of ξ (corresponding to the minimum in Fig. 2) becomes visible.

Summarizing, we have thus shown that equations (1), which are phenomenological, but motivated by clear physical processes, indeed show an instability which is consistent with some essential features of ripple formation. It is worth noting that our analysis, which concentrated on the linearized system in the vicinity of the stationary solution, is universal in the sense that a whole class of models behaves in an analogous way (with some possible redefinition of the coefficients). For example, a non-linear dependence of the velocity V on R does not modify the above analysis, up to a redefinition of V. Note also that all phenomenological coefficients are, at least in principle, measurable in situations independent from ripple formation (since they are diffusion constants, convection velocities, deposition rates *etc.*). In this sense it should be possible to check experimentally for the consistency of the above description.

Our conclusions are very similar to those reached by Anderson [3], on the basis of a simplified model where the flowing phase (what we have called R above) is assumed to be in equilibrium from the outset, and where a rather arbitrary distinction is made between 'saltating grains' which are never captured by the bed, and 'reptating' grains which are captured after exactly one jump. Correspondingly, the structure of the dispersion relations differ in the two approaches. Furthermore, it is difficult to extend Anderson's model beyond the linear instability analysis while our model, in principle, can account for non-linear effects [10].

Finally, there are several open questions which we would like to mention and leave for future work: (i) Can one establish some precise relations between the 'microscopic' coefficients (like wind velocity, polydispersity, elasticity *etc.*) and the phenomenological parameters? (ii) How are the above results modified if one considers two spatial dimensions? Is there an instability corresponding to the transverse wavelike shape of the ripples known from field observation? (iii) What is the ripple shape and height predicted from a non-linear analysis of the equations? (iv) Is there a logarithmic increase of the wavelength in the non-linear regime as reported in [11,12]? (v) Is it important to consider a non-local convection term, rather than the hydrodynamical form written in (1)? The question arises since the relevant wavelength is precisely of the same order as (and not much larger than) the jump length ξ.

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